Pre-service Primary Teachers' Choice of Mathematical Examples: Formative Analysis of Lesson Plan Data

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The learning and teaching of mathematics are key elements for primary school teachers, and various approaches for teaching mathematics to pre-service teachers are evident in mathematics education. This paper reports on a project to develop a critical approach to using mathematical subject knowledge in choosing learning examples and examines data from United Kingdom pre-service teachers' lessons. The data suggests pre-service teachers have no structured method for choosing examples, which impacts on the quality of the learning experiences of students.

In the field of primary mathematics education, recent studies have focussed on the development of teachers' subject knowledge (Shulman, 1986; Rowland, Thwaites, & Huckstep, 2003). This is a complex area and the purpose of this paper is to discuss the issue of choosing examples, a pedagogically critical aspect that depends on the teacher's level of pedagogical content knowledge, and argue that improving mathematical knowledge can enhance the choice and use of such examples by specifically focusing on evidence from four pre-service teachers who formed part of a recent study. This involved the examination of planning documentation for primary mathematics from pre-service teachers on their final school placement, and their consideration of the examples they use in their teaching. The research was based on a naturalistic paradigm with a phenomenological orientation whereby it examined multiple interpretations of different pre-service teachers and how they constructed their own approaches to choosing examples through being immersed in a real teaching context within different school settings. The particular focus was on the way pre-service teachers use theory and teaching resources to identify and select appropriate examples for their teaching.

Seven categories of teachers' knowledge for classroom practice were conceptualised in Shulman's (1986) seminal work that has been used widely in developing approaches to improving and assessing this knowledge. Three of Shulman's categories focus directly on what can be called *content* knowledge, these being *subject matter knowledge* (SMK), *pedagogical content knowledge* (PCK) and *curricular knowledge*. Shulman remarked "a century ago the defining characteristic of pedagogical accomplishment was knowledge of content." This supports Goldsmith's (c.1926) poetic description of an eighteenth century Irish schoolmaster; "And still they gazed, and still the wonder grew, that one small head could carry all he knew."

The phrase 'teachers' knowledge' gives the impression that the teacher was the fount of all knowledge, and certainly of greater knowledge than the students (and their parents). However, it fails to distinguish between the types of knowledge that exist, and their relative importance in terms of what a teacher needs to know, indeed what is sufficient to know and what is necessary to know in order to teach mathematics effectively. The belief that a certain kind of mathematics subject knowledge is needed for teaching and the possibility that this can be assessed has been researched in small-scale studies (for example, Corcoran, 2005).

In V. Steinle, L. Ball & C. Bardini (Eds.), Mathematics education: Yesterday, today and tomorrow (Proceedings of the 36th annual conference of the Mathematics Education Research Group of Australasia). Melbourne, VIC: MERGA.

Pedagogical content knowledge is for Shulman "… the ways of representing the subject which makes it comprehensible to others … [it] also includes an understanding of what makes the learning of specific topics easy or difficult." (1986, p.9) PCK is essentially a conceptualisation of the link between knowing something for oneself and enabling others to know it. Aubrey (1994) defined pedagogical subject knowledge as including "… knowing what knowledge, concepts and strategies children bring to learning, their misconceptions as well as their understandings, and the stages through which they pass towards mastery of topics within a subject area." (p.106)

Ball (1990) echoes Shulman's constructs of substantive and syntactic knowledge in any discipline by making a distinction between knowledge *of* mathematics (meanings underlying procedures) and knowledge *about* mathematics (what makes something true or reasonable in mathematics). Ball found in studying pre-service teachers that, for example, they had significant difficulties with their understanding of the *meaning* of division by fractions. Most could do the calculations, but their explanations were rule-bound, with a reliance on memorising rather than conceptual understanding. More recently, Ball, Thames and Phelps (2008) developed and refined the notion of PCK by studying mathematics teaching and identifying mathematical knowledge *for* teaching based on analyses of mathematical problems that arose during teaching. Their research suggested two subdomains of PCK, *knowledge of content and students,* and *knowledge of content and teaching*, as well as a new subdomain of *specialised content knowledge*, distinct from *common content knowledge* as used by both teachers and non-teachers. Each of these subdomains will impact on choice of examples and hence are significant for pre-service teachers' subject knowledge development. Ball et al (2008) argue that 'when choosing an example, teachers need to predict what students will find interesting and motivating.' (p.397) Improving knowledge of contents and students, they suggest, will lead to better selections of mathematical examples.

The requirements of pre-service teachers in terms of their mathematical knowledge needs is seen as essential to their careers as primary school teachers. Ponte and Chapman (2008, p.226) stated that whilst "… strong knowledge of mathematics does not guarantee that one will be an effective mathematics teacher, teachers who do not have such knowledge are likely to be limited in their ability to help students develop relational and conceptual understanding." Skemp (1976) described *relational understanding* of mathematics as "knowing what to do and why", (p. 21) as opposed to *instrumental understanding* which merely focuses on knowing what to do. This relational understanding is directly related to the conceptual underpinnings of the mathematics, both for primary school students as well as pre-service teachers.

Rowland, Martyn, Barber and Heal (2003) worked on the SKIMA (Subject Knowledge in Mathematics) project to examine elements of teacher knowledge, and this led to the development of the Knowledge Quartet (KQ) (Rowland, Huckstep and Thwaites, 2003; Rowland, Turner, Thwaites and Huckstep, 2009). The KQ categorises events in mathematics lessons with particular reference to the subject matter being taught, and the mathematics-related knowledge that teachers call upon. While Shulman's (1986) distinction between subject matter knowledge and pedagogical knowledge underpins this consideration of mathematics teaching, the Knowledge Quartet (KQ) identifies situations in which such knowledge can be seen in the act of teaching, in the context of a one-year graduate course for primary pre-service teachers.

The four dimensions of the KQ developed from knowledge and beliefs evidenced in mathematics teaching and can be seen in four dimensions named *foundation,* *transformation, connection* and *contingency*. Foundation consists of the knowledge and beliefs learned in the academy, informing pedagogical choices in preparation for teaching. Transformation is about knowledge-in-action, demonstrated through planning and teaching and includes choosing examples. Connection is concerned with ways of linking concepts and structures across discrete mathematics topics, and contingency caters for the 'unplanned' incidents within a lesson, responding to students' ideas and questions, deviating from the lesson agenda and using good constructivist practice to enhance learning from the opportunistic moments that arise.

Classroom examples might be described as falling in one of two main categories, firstly as inductive objects from which the student is provided with an example *of* something. This something is usually general in nature, for example the notion of static angle, or the fact that multiples of 5 end in 0 or 5. With such examples we use particular instances of the generality and this is commonplace in pedagogical practice to embody abstract concepts. An example of this might be the use of 43 – 27 in demonstrating column subtraction. In such an example, the digits 4, 3, 2, and 7 are carefully selected from a choice of options to ensure the appropriate learning related to subtraction procedures and understanding.

Marton and Booth (1997) describe the range of choices of digits open to the teacher in this example as the *dimensions of variation*, an idea which describes the range of affordances and constraints in a given task or activity. The affordances are generally demonstrated in the range of mathematical responses given by students, whilst the constraints within a task filter the students' learning towards a specific objective. In the case of broad mathematical concepts, examples can provoke or facilitate abstraction. Once a set of examples is unified by concept formation for the learner, subsequent examples can be assimilated by the concept, even when they fall outside the student's experience.

In the second category, examples are more often called *exercises*, and rather than being inductive, they are practice-oriented. Exercises tend to be examples drawn from a class of possible such examples. For two-digit subtraction, an exercise may consist of 20 examples drawn from approximately 4000 possible cases, taking all possible permutations of 2-digit subtractions using digits 0-9. Such a set of examples is designed to rehearse a procedure for retention and fluency and is often used for assessment purposes. Further consideration might include the grading of examples from easier to harder, as well as exposing the student to a range of possible situations they might encounter later in life.

The notion of example spaces as a way of helping to define subject knowledge for teaching mathematics is a useful one which can develop and build upon the work of, for example, Watson and Mason (2005) and Zazkis and Leikin (2007). Primary students' learning in mathematics is likely to be influenced by the examples they encounter, and the particular concepts they need to develop are linked to their own example spaces, some of which are generated by the teacher and others by the students themselves. The examples that pre-service teachers generate often highlight underlying misconceptions or inadequacies in subject knowledge.

Methodology

This paper reports on a research project developed from analysis of mathematics lesson plans produced by final year undergraduate pre-service primary teachers following a three-year degree course for teacher education at a UK university. It was suggested that pre-service teachers' mathematical SMK, along with an awareness of relevant PCK determines their choice of examples in lessons, to be addressed through the following research questions:

- What pedagogic considerations do a cohort of pre-service primary teachers use when choosing mathematical examples in the classroom?
- How do these pedagogic considerations fit within current theoretical frameworks in primary mathematics pedagogy?
- Is there a relationship between the cohort of pre-service primary teachers' level of mathematical subject knowledge and the types of examples they select?

One weakness of research into pre-service teachers' knowledge and practice has been the small sample size necessitated by the intensive nature of a qualitative and partially biographical approach and the assumption that teacher education courses can be considered as a constant. Details about the sample selection may prompt questions about validity, but the findings have some claim on generalizability since the data has been embedded in existing literature, not just a selection of cases that demonstrate the mathematical knowledge implications within the research sample taken from one UK university cohort. It is perceivable that the findings might be replicated in other cohorts of pre-service primary teachers, both nationally in the UK and internationally.

Lesson plans were requested from the entire cohort of 112 pre-service teachers and in response 22 pre-service teachers supplied a total of 406 mathematics lessons across the primary age range of 5-11 years in the UK. Within the sample group, there are 18 females and 4 males, which bear close resemblance to the gender proportions found in the primary school teaching population of the UK. For the purpose of analysis, the lesson plans were grouped by year and topic, with the most represented years and topics providing the first layer of specific analysis, followed by a more general analysis of the entire set of lessons to look for extended similarities in terms of the choices of examples by the research sample.

The first level of analysis was therefore carried out using 63 lesson plans for Year 3 and 4 students (ages 7-9 years), specifically on the topic of number, which includes addition, subtraction, multiplication and division, and also fractions, decimals, percentage and ratio (FDPR). These plans were produced by eight (six female, two male) of the 22 pre-service teachers in the sample.

In the following section, I present a number of inductive examples drawn from the collected lesson plans, and identify the use of the examples within two of the four categories defined by Rowland (2008). He distinguished four aspects of awareness when choosing examples, namely *variables, sequencing, representations* and *learning objectives*, and it is the first two of these (variables and sequencing) that I focus on, due to the range of data available from the collected lesson plans on these aspects.

Results

In the first set of examples, exploring the variables category, the choice of numbers in calculations could cause confusion in terms of the task being demonstrated. Using the notion of dimensions of variation mentioned earlier, each example can be considered to be made up of a number of variables, for example when multiplying two numbers there are the two factors and the resulting product. Careful choice of these variables is necessary in order to prevent the learning point being obscured by the variables selected.

Table 1
Characteristics of Participants

Characteristic	Jane	Sharon	Carrie	Lee
GCSE	A	A*	C	C
A-level	C	A	-	-
Degree	2:1	1	3	3
Students	Y4	Y4	Y3/4	Y3/4

a – General Certificate of Secondary Education (GCSE) is the English examination at age 16, 'A-level' is taken at age 18, 'Degree' shows the final undergraduate classification and 'students' refers to the age of the students taught by the pre-service teachers in their final school placement.

One pre-service teacher in the sample group, Jane, is a capable mathematician. Her level of mathematics is one of the highest in the cohort and sample group and I chose to consider her examples to look for evidence that high mathematical attainment, representing good mathematical knowledge, might translate into good pedagogic examples. Her final school placement was with a Year 4 class (age 8-9 years) in a suburban school, and her examples are taken from work on calculations. Firstly her lesson plan showed an example on place value, where Jane was attempting to teach the students the value of the digits in different places or columns. The number she chose was 767, and in asking the students what each of the digits represent, she was including a 7 in both the hundreds and the units place. Given that she had already used a 7 and a 6 to fill the hundreds and tens, she still had a choice of seven other non-zero digits to fill the units column, missing the opportunity to use variation in her choice, as outlined by Marton and Booth (1997).

A later example from Jane's teaching showed her teaching division, using as her objective "to solve simple division problems (TU ÷ U)". Jane makes poor choices when she uses the contextual example of "25 lollies in a jar, how many in 5 jars?" and the non-contextual example '16 divided by 4' a little later in her teaching. With the lollies example, the divisor of 5 is the same as the quotient, and the non-contextual example also has the same divisor and quotient, which could be a source of misconception for the students, as described by Rowland (2008).

It is perhaps conceivable that a pre-service teacher with a higher level of mathematics attainment might not make such mistakes when selecting examples, but the case of Sharon further suggests that this is not necessarily so. Her Year 4 lesson plans reveal the following two examples: 22 – 11 and 50 – 25, both of which provide a result the same as the number being subtracted, suggesting she was unable to account for the confusion that could happen through poor choice of variables (Rowland, 2008).

The second category that provides evidence of pre-service teachers' lack of pedagogic understanding about choice of examples is that of sequencing. Here, I refer to examples where pre-service teachers set out number series and require their students to identify the link between each term in the series and the following term, and use that information to predict or calculate the next one or more terms in the sequence.

Carrie, a pre-service teacher working with a Year 3/4 class (age 7-9 years), has average attainment in mathematics. The lesson plans examined from Carrie's placement demonstrate that she is seeking to engage the students with the notion of sequences in different units of metric measure. Her first example offers an ascending series of lengths, increasing each time by 10cm:

1. 1.23m, 1.33m, 1.43m, 1.53m, …

This example seems relatively straightforward and students at Year 3/4 level should be able to continue this sequence with 1.63m, 1.73m, and so on. It would be interesting to see whether the step from 1.93m to 2.03m caused problems for any student, as it is possible that misconceptions over place value could produce a result of 1.103m.

Carrie's next example provides a more challenging situation, with a sequence that appears to be numerically descending, but on closer inspection the change of numbers and units combine to generate a sequence, all of whose terms have the same length:

1. 10cm, 0.1m, 1/10m, …

It is an interesting task to try to ascertain which direction Carrie was intending the sequence to go from here, as terms 1 and 2 are equal lengths but expressed with different numbers and units, while terms 2 and 3 have equal value in units and numerically. It seems there is no sequence, and the activity is merely to find further equivalences, such as 1/10,000km or 100mm. This is the second example on the lesson plan, and following this there is a return to something rather easier:

1. 25cm, 50cm, 75cm, 100cm, …

In this example, the units remain constant in centimetres, while the numerical values form an arithmetic progression with a common difference of 25. This example is considerably easier than the previous one, which leads one to question Carrie's purpose in providing the second example that is out of step with those either side of it. However, after an example demonstrating sequences quite well, the next provides a novel leap of intellectual challenge by developing a sequence with an arithmetic progression in its numerical value, but the units alternating between centimetres and metres:

1. 10mm, 2cm, 30mm, 4cm, 50mm, …

Such an example is unlikely to support understanding of sequences unless the students are secure with the notion of conversion between the two units of metric measurement. Carrie then offers further examples demonstrating simple arithmetic progressions, before a return to another example of a sequence of lengths with alternating units, similar to that above:

1. 10m, 20m, 30m, 40m, 50m, …
2. 12cm, 14cm, 16cm, 18cm, 20cm, …
3. 100mm, 20cm, 300mm, 40cm, 500mm, …

Looking at the repertoire of Carrie's lessons, it might be interpreted that she lacks the specialised content knowledge described by Ball et al (2008) and possibly a secure knowledge of content and students. Whilst attempting to develop sequences of number patterns, Carrie resorts to varying a dimension of the context that draws students' thinking away from number to the units being used (Marton and Booth, 1997).

By way of comparison, Lee also produced a lesson on sequences, which will be examined now. Lee's plans demonstrated a variety of sequences that afforded possibilities of mathematical thinking and discussion, which perhaps Carrie's did not, suggesting Lee had a greater depth of knowledge for teaching (Ball et al, 2008), in particular a knowledge of content and students, as well as the specialised knowledge which enabled him to choose better examples. Lee's first example was:

Cover every 3rd square on a 100 square.

This is not about continuing a listed sequence but gives the students the experience of practically generating a sequence from a rule, which produces a number pattern for them to explore. This seems a more productive way of exploring number patterns than to be given a pattern for which the rule needs to be determined. Lee's second example developed this idea of generating a sequence on a familiar resource:

Start at 3, count on in 5s to 53

In these examples, Lee enables the students to see the underlying pattern as the sequence is generated, rather than ask them to search for the pattern. This seems a sensible way to proceed as it develops their understanding of sequences for when they face more challenging examples such as those in Carrie's lesson. Lee's next sequence has its earlier terms missing and he uses a common difference of 3 between the terms, the same difference as in the first generated example, providing a link to prior learning:

1. …, …, …, 59, 62, 65

The next example offers the idea of a sequence through a different approach. Lee uses a series of sums, whose answers generate a sequence, common difference 3:

1. 1+2+3=…, 2+3+4=…, …

This shows insight and provides evidence of a connectionist approach (Askew et al, 1997) by offering a varied range of connected mathematical thinking opportunities to generate a single concept, namely sequences. Lee also exhibits here the kinds of choice that Rowland et al (2003) would consider as evidence of using a transformation approach in his use of subject matter knowledge. From this, the learning is challenged in a more unusual way, by offering a final example of fractions written on a strip of paper:

1. 1/2, 1/3, 1/4, 1/5, 1/6, …

This has the initial appearance of a simple sequence, with the denominators increasing by 1 from each term to the next, but the visual effect of plotting these on a paper strip is to demonstrate a sequence whose terms (but not whose sum) converge to a fixed point, and whose differences are always decreasing. This is the last example in what Zazkis and Leikin (2007) might consider a rich example space, varying between routine and non-routine examples, and ones that demonstrate fluency in their variety to develop the relevant concept image.

Reviewing the four pre-service teachers and the examples in their lesson plans, it appears that there is some justification in asserting that personal performance in mathematics may be linked to the quality of mathematical examples selected for teaching, although this differs slightly from Ball et al (2008) in their view that better examples come from enhanced knowledge of content and student. Both factors could be seen as significant and worthy of consideration by teacher educators in seeking to improve learning opportunities for pre-service teachers, perhaps ensuring more time can be spent assessing students' abilities during the early part of school placements.

Conclusion

This study sought to consider how pre-service teachers plan their mathematics lessons using a range of sources, such as textbooks, student workbooks, advice from serving teachers, and so on. With regard to the research questions of the study, the evidence suggests that whilst all pre-service teachers are made aware of the UK's Primary National Strategy for Literacy and Mathematics (2006) as part of their training, the extent to which they use it in school varies from using it as a major support and guidance for their planning to referring to it as useful guidance but not exclusively, preferring to draw from a range of other resources or their own levels of subject knowledge where they are confident that these are suitably high. However, perhaps the over-riding feature demonstrated by the research data in relation to this question is that the seven case study pre-service teachers, from a range of mathematical attainments and backgrounds, were not at all clear about what constitutes a mathematical 'example'. Therefore, any discussion about their choice of examples for mathematical learning was limited to their interpretation of examples, which was usually vague. Evidence from the data points to the interpretation that pre-service primary teachers are aware of theoretical frameworks whilst completing their course, but do not make significant connections between theory and classroom practice and do not, on the whole, use theoretical notions to aid their planning or choices of examples.

Whilst each of the pre-service teachers identifies with the idea that subject knowledge is related to the choice of examples, their views on what subject knowledge includes and their understanding of 'examples' leads to a blurred interpretation of that relationship, which links with the findings for the first research question. This has implications for teacher education courses in terms of how pre-service teachers are taught to choose appropriate examples, as identified by Rowland's (2008) categories.

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